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Non-Fourier heat conduction in a finite medium under periodic surface thermal disturbance

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Abstract—In this paper, the non-Fourier effect in a slab subjected to a periodic thermal disturbance is investigated by deriving the analytical solution of the hyperbolic heat conduction equation. The temperature profiles at the front and rear surfaces of the slab are calculated for various relaxation time. By comparing the results with those obtained from the Fourier parabolic heat conduction equation, a transition condition between the 'parabolic' and 'hyperbolic' behavior at both surfaces is obtained. The phase and amplitude difference between the front and the rear surface is calculated numerically as a function of the relaxation Fourier number and the results are shown graphically.

INTRODUCTION

The non-Fourier effect becomes more and more attractive in practical engineering problems because the use of heat sources such as laser and microwave with extremely short duration or very high frequency has found numerous applications for purposes such as surface melting of metal [1] and sintering of ceramics [2]. In such situations, the classical Fourier's heat diffusion theory will become inaccurate.

Theoretically, the Fourier's heat-conduction equation leads to the solutions exhibiting infinite propagation speed of thermal signals. In order to eliminate this paradox, Cattaneo [3] and Vernotte [4] independently postulated a time-dependent relaxation model for the heat flux in solids:

$$q = -\lambda \nabla T - \tau_0 \frac{\partial}{\partial t} q \qquad (1)$$

where q is the heat-flux, T is the temperature, λ is the thermal conductivity and τ_0 is the relaxation time. Equation (1) and the conservation equation of energy lead to a description of an unsteady temperature profile in the form of the hyperbolic equation

$$a\nabla^2 T = \frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} \qquad (2)$$

where a is the thermal diffusivity, $\sqrt{(a/\tau_0)}$ is the propagation speed of temperature wave.

Various solutions of the hyperbolic heat conduction equation for finite medium under different initial and boundary conditions can be found in literature, which are listed as following: (1) in 1972, Taitel gave an analytical solution for a thin layer subjected to a step change of temperature on its both sides [5]; (2) in 1982, Garey gave a numerical solution for a thin layer subjected to a step change of temperature on one side [6]; (3) in 1984, Ozisik gave an analytical solution in a finite slab with insulated boundaries [7], in his treatment, a volumetric energy source was used; (4) in 1985, Frankel, using flux formulation of hyperbolic heat conduction equation, gave an analytical solution for a finite slab under boundary condition of rectangular heat pulse [8]; (5) in 1985 and 1986, Glass gave numerical solutions for a finite medium with surface radiation [9] and temperature-dependent conductivity [10], respectively and (6) in 1988, Gembarovic gave an analytical solution for a finite slab under boundary condition of instantaneous heat pulse and extended heat pulse [11]. Most previous works were performed for a pulse heat flux or a sudden temperature change. But for a periodic flux in a finite medium, the work is seldom found in literature.

The present work investigates analytically the non-Fourier effects in a finite medium subjected to a periodic heat flux condition by using the hyperbolic heat conduction model. The temperature profiles at the front and rear surfaces are calculated for different $\tau_0\omega$, the results are compared with those obtained from Fourier parabolic heat conduction equation. The 'transition condition' between the 'parabolic' and 'hyperbolic' behavior of temperature response at both of the surfaces are obtained. The non-Fourier effect is discussed by comparing the phase and amplitude between the front and rear surfaces.

ANALYSIS

Consider a slab as a finite medium with the thickness of L with insulated boundaries in which one-dimensional heat conduction and constant thermal

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NOMENCLATURE

A	amplitude of periodic temperature response
a	thermal diffusivity
c	specific heat capacity
Fo	Fourier number, at/L^2
Fo_1	Fourier number based on frequency of heating flux, $a/\omega L^2$
L	thickness of medium
$L[\,]$	Laplace transformation
$L^{-1}[\,]$	inverse Laplace transformation
q	heat flux
$\bar{q}$	Laplace transformation of q
Re[ ]	residue
s	Laplace variable
T	temperature
$\bar{T}$	Laplace transformation of T
t	time
V	dimensionless temperature defined by equation (16)
Ve	Vernotte number, $\sqrt{(a\tau_0)}/L$
X	dimensionless spatial variable, x/L
x	spatial variable.

Greek symbols

φ	phase of periodic temperature response
$\Delta\phi$	phase difference, $(\varphi_F-\varphi_R)/2\pi Fo_1$
λ	thermal conductivity
ρ	mass density
τ_0	relaxation time
ω	frequency of periodic heating flux.

Subscripts

F	front surface
R	rear surface.

properties prevail. The medium is initially in equilibrium at temperature $T(x, 0) = 0$, from time $t = 0$ the external surface at $x = 0$ is exposed to a periodic heat flux with the amplitude q_0 and the frequency ω. In this situation, the general non-Fourier heat conduction, equation (2), should be in one-dimensional form

$$a\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} + \tau_0\frac{\partial^2 T}{\partial t^2}. \quad (3)$$

The boundary and the initial conditions are

$$-\lambda\frac{\partial}{\partial x}T(0,t) = \tau_0\frac{\partial}{\partial t}q(0,t)+q(0,t), \quad q(0,t) = q_0\cos\omega t \quad (4a)$$

$$\frac{\partial}{\partial x}T(L,t) = 0 \quad (4b)$$

$$T(x,0) = 0, \quad \frac{\partial}{\partial t}T(x,0) = 0, \quad q(x,0) = 0. \quad (5)$$

If the Laplace transformation is applied to equation (3) by taking into account the initial conditions (5), the following subsidiary equation is obtained as

$$a\bar{T}'' - (s-\tau_0 s^2)\bar{T} = 0 \quad (6)$$

with the conditions

$$\frac{\partial}{\partial x}\bar{T}(0,s) = -\frac{q_0}{\lambda}(1+\tau_0 s)\bar{q}(s) \quad \text{and} \quad \frac{\partial}{\partial x}\bar{T}(L,s) = 0 \quad (7)$$

where $\bar{T}(x, s) = L[T(x, t)]$ and $\bar{q}(s) = (1/q_0)L[q(0, t)]$ are the Laplace transform of $T(x, t)$ and $q(0, t)$, respectively.

The solution of equation (6) with respect to the condition of equation (7) is

$$\bar{T}(x,s) = \frac{q_0}{\rho c L}\frac{L\sqrt{(1+\tau_0 s)/a}\cosh r(x-L)}{\sqrt{s}\sinh rL}\bar{q}(s) \quad (8)$$

where $r = \sqrt{(s+\tau_0 s^2)/a}$, ρ is the density, c is the specific heat capacity. For convenience in subsequent derivation, the following functions are introduced as

$$F_1(s) = \frac{L\sqrt{(1+\tau_0 s)/a}\cosh r(x-L)}{\sqrt{s}\sinh rL} \quad F_2(s) = \bar{q}(s)$$

$$\text{and} \quad F(s) = F_1(s)F_2(s) \quad (9)$$

and

$$f(x,t) = L^{-1}[F(s)] \quad f_1(x,t) = L^{-1}[F_1(s)]$$

$$f_2(x,t) = L^{-1}[F_2(s)]. \quad (10)$$

Then the inverse transformation of equation (8) can be expressed as

$$T(x,t) = L^{-1}[\bar{T}(x,s)] = \frac{q_0}{\rho c L}L^{-1}[F(s)] = \frac{q_0}{\rho c L}f(x,t). \quad (11)$$

Equations (9) and (10) along with the property of Laplace transform give

$$f(x,t) = f_1(x,t)*f_2(x,t) = \int_0^t f_1(x,t')f_2(x,t-t')\,\mathrm{d}t' \tag{12}$$

where $f_1(x,t)$ is derived in the Appendix and $f_2(x,t)$ is determined by equations (9) and (10), the results are given as follows:

$$f_1(x,t) = \begin{cases} 1+\sum_{n=1}^{N}\cos\left(\frac{n\pi x}{L}\right)\left[\frac{1+\beta}{\beta}\mathrm{e}^{-\frac{t}{2\tau_0}(1-\beta)} - \frac{1-\beta}{\beta}\mathrm{e}^{-\frac{t}{2\tau_0}(1+\beta)}\right], & \text{when } \beta = \text{real} \\ 1+2\sum_{n=N}^{\infty}\cos\left(\frac{n\pi x}{L}\right)\mathrm{e}^{-\frac{t}{2\tau_0}}\left[\cos\frac{\beta_1 t}{2\tau_0} + \frac{1}{\beta_1}\sin\frac{\beta_1 t}{2\tau_0}\right] & \text{when } \beta = \beta_1 i \end{cases} \tag{13}$$

$$f_2(x,t) = L^{-1}[\bar{q}(s)] = \frac{q(0,t)}{q_0} = \cos\omega t \tag{14}$$

where β is defined by equation (A6) in the Appendix, N is a point at which β changes from a real number to a complex number with increasing the value of n. Substituting equations (13) and (14) into equation (12) yields the following,

$$f(x,t) = \int_0^t\left\{1+\sum_{n=1}^{N}\cos\left(\frac{n\pi x}{L}\right)\left[\frac{1+\beta}{\beta}\mathrm{e}^{-\frac{t'}{2\tau_0}(1-\beta)} - \frac{1-\beta}{\beta}\mathrm{e}^{-\frac{t'}{2\tau_0}(1+\beta)}\right]\right\}\cos\omega(t-t')\,\mathrm{d}t', \quad \text{when } \beta = \text{real} \tag{15a}$$

$$f(x,t) = \int_0^t\left\{1+2\sum_{n=N}^{\infty}\cos\left(\frac{n\pi x}{L}\right)\mathrm{e}^{-\frac{t'}{2\tau_0}}\left[\cos\frac{\beta_1 t'}{2\tau_0} + \frac{1}{\beta_1}\sin\frac{\beta_1 t'}{2\pi_0}\right]\right\}\cos\omega(t-t')\,\mathrm{d}t', \quad \text{when } \beta = \beta_1 i. \tag{15b}$$

By a straightforward series of manipulations of integration, the non-Fourier temperature wave inside the finite medium should be obtained. By introducing the following dimensionless quantities

$$V(X,Fo) = \frac{T(X,Fo)}{q_0/\rho c\omega L} \quad Fo = \frac{at}{L^2}$$

$$Fo_1 = \frac{a}{\omega L^2} \quad Ve^2 = \frac{a\tau_0}{L^2} \quad X = \frac{x}{L} \tag{16}$$

where Fo is Fourier number, Ve is Vernotte number which is the relaxation Fourier number and $1/Ve = (L/a)\sqrt{(a/\tau_0)}$ denotes the dimensionless speed of propagation of temperature wave, and X is the dimensionless coordinate, the temperature profile inside the finite medium is obtained as

$$V(X,Fo) = \sin\frac{Fo}{Fo_1} + \sum_{n=1}^{N}\cos(n\pi X) \times \left\{\begin{aligned} &\frac{(1+\beta)/\beta}{\zeta_-^2+1}\left(\zeta_-\cos\frac{Fo}{Fo_1}+\sin\frac{Fo}{Fo_1}-\zeta_-\mathrm{e}^{-\zeta_-\frac{Fo}{Fo_1}}\right) \\ &-\frac{(1-\beta)/\beta}{\zeta_+^2+1}\left(\zeta_+\cos\frac{Fo}{Fo_1}+\sin\frac{Fo}{Fo_1}-\zeta_+\mathrm{e}^{-\zeta_+\frac{Fo}{Fo_1}}\right)\end{aligned}\right.$$

$$\text{when } \beta = \text{real} \tag{17a}$$

and

$$V(X,Fo) = \sin\frac{Fo}{Fo_1} + \frac{2Ve^2}{Fo_1}\sum_{n=N}^{\infty}\cos(n\pi X) \times \left\{\begin{aligned} &\frac{\cos\frac{Fo}{Fo_1}-\frac{1}{\beta_1}\sin\frac{Fo}{Fo_1}}{1+\xi_+^2}\left[1+\mathrm{e}^{-\frac{Fo}{2Ve^2}}\left(\xi_+\sin\frac{\xi_+Fo}{2Ve^2}-\cos\frac{\xi_+Fo}{2Ve^2}\right)\right] \\ &+\frac{\cos\frac{Fo}{Fo_1}+\frac{1}{\beta_1}\sin\frac{Fo}{Fo_1}}{1+\xi_-^2}\left[1+\mathrm{e}^{-\frac{Fo}{2Ve^2}}\left(\xi_-\sin\frac{\xi_-Fo}{2Ve^2}-\cos\frac{\xi_-Fo}{2Ve^2}\right)\right] \\ &+\frac{\sin\frac{Fo}{Fo_1}+\frac{1}{\beta_1}\cos\frac{Fo}{Fo_1}}{1+\xi_+^2}\left[\xi_+-\mathrm{e}^{-\frac{Fo}{2Ve^2}}\left(\sin\frac{\xi_+Fo}{2Ve^2}+\xi_+\cos\frac{\xi_+Fo}{2Ve^2}\right)\right] \\ &-\frac{\sin\frac{Fo}{Fo_1}-\frac{1}{\beta_1}\cos\frac{Fo}{Fo_1}}{1+\xi_-^2}\left[\xi_--\mathrm{e}^{-\frac{Fo}{2Ve^2}}\left(\sin\frac{\xi_-Fo}{2Ve^2}+\xi_-\cos\frac{\xi_-Fo}{2Ve^2}\right)\right]\end{aligned}\right.$$

$$\text{when } \beta = \beta_1 i \tag{17b}$$

where

$$\zeta_\pm = (1\pm\beta)\frac{Fo_1}{2Ve^2} \quad \xi_\pm = \beta_1 \pm \frac{2Ve^2}{Fo_1}. \tag{18}$$

For checking the reasonability of the solution, we consider a limit situation of above solution, $\tau_0 \to 0$, i.e. $Ve \to 0$, the non-Fourier solution should go back to the Fourier solution. Under this condition, we have

$$\zeta_- = (1-\sqrt{1-(2n\pi Ve)^2})\frac{Fo_1}{2Ve^2}$$
$$= [1-(1-2n^2\pi^2 Ve^2+2n^2\pi^2 Ve^2 O(Ve^2))]\frac{Fo_1}{2Ve^2}$$
$$= n^2\pi^2 Fo_1 \quad (19a)$$

and

$$(1+\beta)/\beta = 2, \quad (1-\beta)/\beta = 0. \quad (19b)$$

Substituting equation (19) into equation (17) yields

$$V(X, Fo) = \sin\frac{Fo}{Fo_1} + 2\sum_{n=1}^{\infty}\frac{\cos(n\pi X)}{(n^2\pi^2 Fo_1)^2+1}\left(n^2\pi^2 Fo_1\cos\frac{Fo}{Fo_1} + \sin\frac{Fo}{Fo_1} - n^2\pi^2 Fo_1\, e^{-n^2\pi^2 Fo}\right). \quad (20)$$

This equation coincides with the Fourier solution which can be easily derived by using a Laplace transformation technique. At first view equation (20) seems to be unable to satisfy the Fourier form of the boundary condition (4a), while actually it is able to in the sense of generalized function [12].

CALCULATED RESULTS AND DISCUSSION

Utilizing equation (17), numerical computation was performed in order to display the temperature profile arising from a periodic surface heat source at $x = 0$ on an infinite slab with the thickness L.

Figure 1 shows the temperature response during an oscillatory surface thermal disturbance with the period $Fo_1 = 0.25$ at the front and rear surfaces ($X = 0$ and 1) and for $Ve = 0.8$. This figure gives a general transient temperature behavior of non-Fourier heat conduction in the slab under periodic surface heating. The series of 'jump points' at both of the curves are the reaching moment of the propagating wavefront of the temperature wave. It can be seen that the wavefront positions are located at $Fo = Ve$, $3Ve$, $5Ve, \ldots$ for the rear surface and $Fo = 2Ve$, $4Ve$, $6Ve, \ldots$ for the front surface. Because the thickness of the slab is 1.0 and the wave propagation speed is $1/Ve$, the propagation takes time Ve from one side of the slab to the other, therefore, the 'jump points' denote the moment at which the thermal wavefront reaches the front or rear surface after propagation through the medium and many times of reflection at the two surfaces. The figure shows the propagation, the 'jump point' gets lower and lower relatively then disappears, and the temperature response becomes a smooth periodic wave.

Figures 2(a) and (b) show the temperature response at both the surfaces respectively after a periodic surface heat flux for the period $Fo_1 = 0.25$ with various values of $Ve^2/Fo_1 = \tau_0\omega$. In both of the figures, the condition $Ve = 0.07$, 0.1, 0.2, 0.3 corresponds to $\tau_0\omega = Ve^2/Fo_1 = 0.02$, 0.04, 0.16, 0.41. The temperature solutions predicted by the Fourier heat conduction equation are also prepared based on equation (20) in order to compare with those from non-Fourier conduction equation. By this way, the transition between 'hyperbolic' and 'parabolic' can be displayed

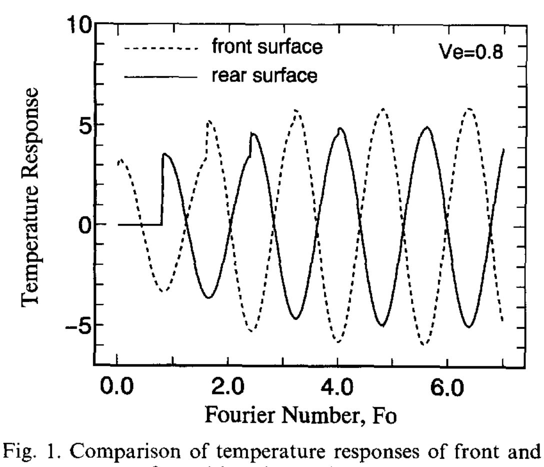

Fig. 1. Comparison of temperature responses of front and rear surface with a given value $Fo_1 = 0.25$.

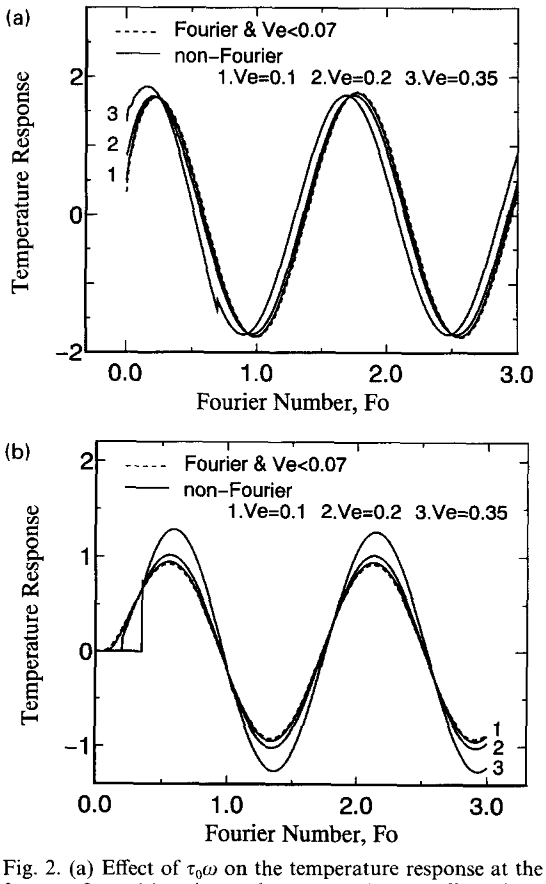

Fig. 2. (a) Effect of $\tau_0\omega$ on the temperature response at the front surface with a given value $Fo_1 = 0.25$. (b) Effect of $\tau_0\omega$ on the temperature response at the rear surface with a given value $Fo_1 = 0.25$.

clearly. The comparison shows that, at both of the surfaces when $Ve \leqslant 0.07$, i.e. $\tau_0\omega \leqslant 0.02$, the difference between the results from the hyperbolic and the parabolic equation is below 1%, i.e. the temperature responses predicted by Fourier and Non-Fourier are almost the same as each other. In other words, the inequality $\tau_0\omega > 0.02$ represents the condition of the occurrence of difference in temperature response between the Fourier and the non-Fourier solution.

For periodic thermal disturbance, it is necessary to give a discussion about the phase and amplitude difference between the front and rear surfaces of the periodic temperature response. Actually this discussion is significant only for small value of Vernotte number, because increasing Ve makes the temperature response deviate from a normal periodic curve (see Fig. 1). It is not easy to give a simple analytical expression of the phase and amplitude difference, so the numerical calculation is performed by means of the temperature response [equation (17)]. Figure 3 shows the phase difference between the front and the rear surface vs Vernotte number for a given period $Fo_1 = 0.25$, in which the Fourier result is also presented. The vertical coordinate is $\Delta\phi = (\varphi_F - \varphi_R)/2\pi Fo_1$, where $(\varphi_F - \varphi_R)$ is determined by the time difference of the first minimum point (peak) in temperature response between the front and rear surfaces. It can be seen that, when $Ve \to 0$, the phase difference of non-Fourier approaches that of the Fourier. With increasing of the Vernotte number, i.e. with enhance of the non-Fourier effect, the phase difference increases. For the same periodic heat flux the Fourier heat conduction predicts a minimum limit of the phase difference. This because non-Fourier law predicts a relative low speed of thermal disturbance propagation.

Figure 4 shows the amplitude damping after propagation through the medium as a function of Vernotte number for a given value $Fo_1 = 0.25$, in which the Fourier results are also present. Here an amplitude damping is defined by $(A_F - A_R)$, where A_F and A_R are the amplitude of the first minimum point of the front and rear surface, respectively. The figure illustrates that, under the condition $Ve \to 0$, the amplitude damping of the non-Fourier form approaches that of

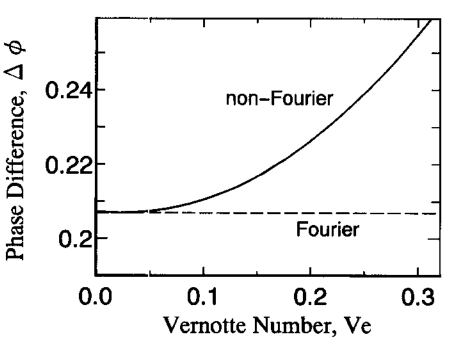

Fig. 3. Phase difference between the front and the rear surface as a function of Vernotte number.

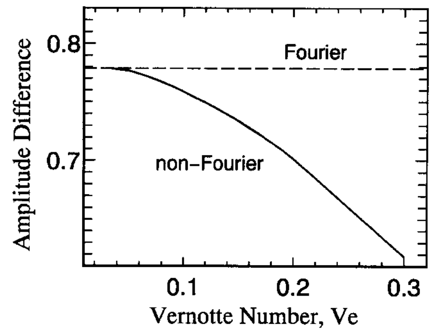

Fig. 4. Amplitude difference between the front and the rear surface as a function of Vernotte number.

the Fourier form, with increasing of Vernotte number, amplitude damping decreases relatively. For the same periodic heat flux the Fourier heat conduction predicts a maximum damping of amplitude. This means that when a periodic heat flux propagates through a medium in a non-Fourier form, the amplitude will have a relative small damping. The stronger the non-Fourier effect is, the smaller the damping in amplitude is.

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APPENDIX

Equation (9) can be expressed as

$$F_1(s)=\frac{A(s)}{B(s)}=\frac{\cosh r(x-L)}{\frac{\sqrt{s}}{L\sqrt{(1+\tau_0 s)/a}}\sinh rL}$$

$$=\frac{1+\frac{1}{2!}\left(\frac{s+\tau_0 s^2}{a}\right)(x-L)^2+\frac{1}{4!}\left(\frac{s+\tau_0 s^2}{a}\right)^2(x-L)^4+\cdots}{s+\frac{1}{3!}s^2L^2\left(\frac{1+\tau_0 s}{a}\right)+\frac{1}{5!}s^3L^4\left(\frac{1+\tau_0 s}{a}\right)^2+\cdots}. \quad \text{(A1)}$$

According to the definition of the inverse Laplace transformation and the residue theorem, the original function of $F_1(s)$ can be expressed as

$$f_1(x,t)=L^{-1}[F_1(s)]=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{st}\frac{A(s)}{B(s)}\mathrm{d}s$$

$$=\sum_{n=1}^{M}\mathrm{Re}\left(\frac{A(s)}{B(s)}e^{st},s_n\right) \quad \text{(A2)}$$

where $s_1,\ldots,s_n$ are all of the poles of $F_1(s)$, Re[] is the notation of residue.

Letting $B(s)=0$, we can obtain all of the poles as following,

$$s_1=0 \quad \text{(A3)}$$

$$s'_n=\begin{cases}-\frac{1}{2\tau_0}(1+\beta) & \text{when } \beta=\text{real}\\ -\frac{1}{2\tau_0}(1+\beta_1 i) & \text{when } \beta=\beta_1 i\end{cases} \quad \text{(A4)}$$

$$s''_n=\begin{cases}-\frac{1}{2\tau_0}(1+\beta) & \text{when } \beta=\text{real}\\ -\frac{1}{2\tau_0}(1+\beta_1 i) & \text{when } \beta=\beta_1 i\end{cases} \quad \text{(A5)}$$

where

$$\beta=\sqrt{1-4n^2\pi^2\tau_0 a/L^2}\quad \beta_1=\sqrt{4n^2\pi^2\tau_0 a/L^2-1}\quad n=1,2,\ldots \quad \text{(A6)}$$

Since all of the poles are of the first order, the residue can be calculated as follows.

(1) For s_1

$$\mathrm{Re}[e^{st}F_1(s),s_1]=\frac{A(s_1)}{B'(s_1)}e^{s_1 t}=1. \quad \text{(A7)}$$

(2) For s'_n

$$Re[e^{st}F_1(s),s'_n]=\frac{A(s'_n)}{B'(s'_n)}e^{s'_n t}$$

$$=\begin{cases}-\frac{1-\beta}{\beta}\cos\left(\frac{n\pi x}{L}\right)\exp\left[-\frac{t}{2\tau_0}(1+\beta)\right] & \text{when } \beta=\text{real}\\ \cos\left(\frac{n\pi x}{L}\right)e^{-\frac{t}{2\tau_0}}\left[\cos\frac{\beta_1 t}{2\tau_0}+\frac{1}{\beta_1}\sin\frac{\beta_1 t}{2\tau_0}\right] & \text{when } \beta=\beta_1 i.\end{cases} \quad \text{(A8)}$$

(3) For s''_n

$$\mathrm{Re}[e^{st}F_1(s),s''_n]=\frac{A(s''_n)}{B'(s''_n)}e^{s''_n t}$$

$$=\begin{cases}\frac{1+\beta}{\beta}\cos\left(\frac{n\pi x}{L}\right)\exp\left[-\frac{t}{2\tau_0}(1-\beta)\right] & \text{when } \beta=\text{real}\\ \cos\left(\frac{n\pi x}{L}\right)e^{-\frac{t}{2\tau_0}}\left[\cos\frac{\beta_1 t}{2\tau_0}+\frac{1}{\beta_1}\sin\frac{\beta_1 t}{2\tau_0}\right] & \text{when } \beta=\beta_1 i.\end{cases} \quad \text{(A9)}$$

From equation (A2) and equations (A7)–(A9), the inverse transformation of $f_1(s)$ is obtained as follows,

$$f_1(x,t)=\mathrm{Re}[e^{st}F_1(s),s_1]+\sum_{n=1}^{\infty}\{\mathrm{Re}[e^{st}F_1(s),s'_n]+\mathrm{Re}[e^{st}F_1(s),s''_n]\} \quad \text{(A10)}$$

and the result is given by equation (13).